Fuzzy BTZ

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Outline:

1. Introduction
2. AdS & BTZ
3. $SL(2, \mathbb{R})$
4. Coordinates, momenta
5. BTZ identifications
6. Radial coordinate
7. Summary

Why?

Why study noncommutative geometry?

- mathematical interest and
- models of quantum spacetime
- many examples in 2d, a few realistic spaces in 4d
- black holes? (singularity, horizons, # of states)
- how to deal with noncommutative time? (non-unitarity, causality)

Why construct noncommutative frames?

- frame formalism is well adjusted to describe spaces with high degree of symmetry, interesting in astrophysics and cosmology
- belief that the model is not exhausted: physically, higher-dimensional spaces; mathematically, representation theory

Talk is based on work with I. Burić, arXiv:2204.03673

NC frames

Noncommutative frame formalism was introduced many times here.

1. basic
   - spacetime $\mathcal{A}$ = abstract algebra or concrete representation
   - coordinates x^μ = hermitian generators of $\mathcal{A}$
   - momenta p_α = generators of 'translations', $e_\alpha f = [p_\alpha, f]$

$$x^\mu x^\nu \neq x^\nu x^\mu, \quad [x^\mu, x^\nu] = i\kern-0.1em\bar{k} J^{\mu\nu}(x)$$

2. geometry
   - frame: set of vector fields e_α that define the tangent space
   - dual 1-forms θ^α freely generate cotangent space $\Omega^1(\mathcal{A})$
   - frame is locally orthonormal, $g(\theta^\alpha \otimes \theta^\beta) = \eta^{\alpha\beta}$

$$[f, \theta^\alpha] = 0$$

NC frames

- "correspondence principle" $[p_\alpha, x^\mu] = e^\mu_\alpha(x)$
- metric $g^{\mu\nu} = e^\mu_\alpha \, e^\nu_\beta \, \eta^{\alpha\beta}$
- differential $df = (e_\alpha f)\,\theta^\alpha \,, \quad dx^\mu = e^\mu_\alpha\,\theta^\alpha, \quad d^2 = 0$
- wedge product $\theta^\alpha \wedge \theta^\beta = P^{\alpha\beta}{}_{\gamma\delta}\,\theta^\gamma \otimes \theta^\delta$
- torsion $\Theta^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta\,\theta^\beta$
- curvature $\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\,\omega^\gamma{}_\beta = \frac{1}{2}\,R^\alpha{}_{\beta\gamma\delta}\,\theta^\gamma\theta^\delta$

$$\theta^\alpha\theta^\beta \neq -\theta^\beta\theta^\alpha$$

3. constraints that follow from compatibility

$$[p_\gamma, p_\delta] = \left(\delta^{\alpha\beta}_{\gamma\delta} - 2P^{\alpha\beta}{}_{\gamma\delta}\right) p_\alpha p_\beta + F^\alpha{}_{\gamma\delta} p_\alpha + K_{\gamma\delta}$$

AdS_3 is a space of maximal symmetry, the symmetry group is $SO(2,2)$.

1. embedding coordinates

$$-v^2 - u^2 + x^2 + y^2 = -\ell^2 = -\frac{1}{\Lambda}$$

$$ds^2 = -dv^2 - du^2 + dx^2 + dy^2$$

2. polar coordinates

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \frac{1}{\frac{r^2}{\ell^2} + 1}\, dr^2 + r^2 d\theta^2 \qquad \theta\,, t/\ell \in (0, 2\pi)$$

$$v = \sqrt{\ell^2 + r^2}\,\cos(t/\ell)\,, \quad u = \sqrt{\ell^2 + r^2}\,\sin(t/\ell)\,, \quad x = r\cos\theta\,, \quad y = r\sin\theta$$

AdS_3 admits closed timelike curves. The universal covering space $\widetilde{\mathrm{AdS}_3}$ is obtained by unwrapping the time: $t \in (-\infty, \infty)$

3. Poincaré coordinates

$$ds^2 = \frac{\ell^2}{z^2}\left(-d\gamma^2 + d\beta^2 + dz^2\right)$$

$$z = \frac{\ell}{u+x}\,, \quad \beta = \frac{y}{u+x}\,, \quad \gamma = -\frac{v}{u+x}\,, \qquad z \in (0,\infty), \ \beta, \gamma \in (-\infty, \infty)$$

Poincaré coordinates cover half of the AdS hyperboloid.

BTZ

BTZ space is a rotating black hole of mass M and angular momentum J.

1. line element

$$ds^2 = -N^2 dt^2 + \frac{1}{N^2}\, dr^2 + r^2 (N^\phi dt + d\phi)^2$$

$$N^2 = \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2}\,, \quad N^\phi = -\frac{J}{2r^2}$$

non-rotating black hole

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \frac{1}{\frac{r^2}{\ell^2} - M}\, dr^2 + r^2 d\theta^2$$

horizons

$$r_+ \pm r_- = \sqrt{M\ell^2 \pm J\ell}$$

2. discrete identifications

BTZ black hole is locally isometric to AdS_3 and can be obtained as a discrete quotient of a subset $\widetilde{AdS'_3}$:

– first identify points under the action of a discrete subgroup of isometries

$$X \to e^{2\pi n \xi'} X\,, \quad n \in \mathbb{Z}$$

generated by vector field ξ', $\xi' = \frac{r_+}{\ell} M_{12} - \frac{r_-}{\ell} M_{03}\,,$

– then cut out spacetime regions $\xi' \cdot \xi' < 0$ (no closed timelike curves).

BTZ has constant curvature and a geodesic singularity.

3. Poincaré coordinates

After discrete identifications, BTZ black hole acquires three types of regions separated by inner and outer horizons. Regions admit local Poincaré coordinates that provide an infinite set of Poincaré patches.

In each of the regions, radial coordinate r is related to z, γ and β as

$$B(r) = \ell^2 \, \frac{r^2 - r_+^2}{r_+^2 - r_-^2} = \ell^2 \, \frac{\beta^2 - \gamma^2}{z^2} \ .$$

BTZ identifications are given by

- in polar coordinates: $t \to t, \ r \to r, \ \phi \to \phi + 2\pi n$
- in Poincaré coordinates:

$$z \mapsto e^{-\frac{2\pi r_+ n}{\ell}} z \,, \quad (\beta - \gamma) \mapsto e^{-\frac{2\pi (r_+ + r_-) n}{\ell}} (\beta - \gamma), \quad (\beta + \gamma) \mapsto e^{-\frac{2\pi (r_+ - r_-) n}{\ell}} (\beta + \gamma)$$

- AdS$_3$ hyperboloid is isometric to $SL(2,\mathbb{R})$,

$$g(X) = \frac{1}{\ell}\begin{pmatrix} u+x & y+v \\ y-v & u-x \end{pmatrix}$$

- so are their isometry groups, $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$
  $g(X)$ transforms as

$$g(X) \to g_1\, g(X)\, g_2^{-1}\,, \quad g_1, g_2 \in SL(2,\mathbb{R})$$

- BTZ identifications are $g(X) \sim \rho_L\, g(X)\, \rho_R$, with $\rho_{L,R}$

$$\rho_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix}, \;\; \rho_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix}$$

$SL(2,\mathbb{R})$

- $\mathfrak{so}(2,2)$ is a direct sum of two $\mathfrak{sl}(2,\mathbb{R})$ subalgebras with generators H, E_+, E_- and $\bar{H}$, $\bar{E}_+$, $\bar{E}_-$,

$$[H, E_+] = E_+, \qquad [H, E_-] = -E_-, \qquad [E_+, E_-] = 2H\,,$$
$$[\bar{H}, \bar{E}_+] = \bar{E}_+, \qquad [\bar{H}, \bar{E}_-] = -\bar{E}_-, \qquad [\bar{E}_+, \bar{E}_-] = 2\bar{H}\ .$$

- $SL(2,\mathbb{R})$ has three series of unitary irreducible representations: principal, discrete and complementary.
- there are two sets of discrete series representations, T_l^- and T_l^+. l is a half integer, in the first case $l \leq -1$. The carrier space of T_l^- is that of analytic functions in the upper half plane $F(z)$, square integrable with respect to

$$(F_1, F_2) = \frac{1}{2\Gamma(-2l-1)} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, y^{-2l-2}\, \overline{F_1(z)} F_2(z)$$

- another realisation of T_l^- is in the Fourier space of functions defined on the semi-axis $(0,\infty)$,

$$F(z)=\int_0^\infty dx\ e^{ixz}\hat{F}(x)\ , \quad (\hat{F}_1,\hat{F}_2)=2^{2l+1}\pi\int_0^\infty dx\ x^{2l+1}\overline{\hat{F}_1(x)}\hat{F}_2(x)$$

- in the Fourier space, generators of $SL(2,\mathbb{R})$ are no longer represented by the first-order differential operators; however

$$H=x\partial_x+l+1\,, \qquad E_+=-ix$$

- the spectrum of iE_+ is $\lambda\in(0,\infty)$ with eigenfunctions $\delta(x-\lambda)$: iE_+ is a positive operator and therefore BTZ coordinates are, in the following, well defined.

Coordinates, momenta

Classical orthonormal frame in Poincaré coordinates is $e^\mu_\alpha = \frac{z}{\ell}\,\delta^\mu_\alpha$

It is quantised by operators $z,\ \beta,\ \gamma,\ p_z,\ p_\beta,\ p_\gamma$ that obey

$$[p_\gamma, \gamma] = \frac{z}{\ell}\,, \quad [p_\beta, \beta] = \frac{z}{\ell}\,, \quad [p_z, z] = \frac{z}{\ell}$$

with all other momentum-coordinate commutators vanishing.

We need to identify momenta and coordinates simultaneously.

Frame relations are satisfied by operators

$$p_z = \frac{1}{\ell}\,(H+\bar H), \qquad z = 2i\,\frac{\ell^2}{\hbar}\; E_+^a\,\bar E_+^{1-a}\,,$$

$$p_\beta = \frac{\ell}{\hbar}\,(E_+ + \bar E_+), \quad \beta = -iE_+^{a-1}\bar E_+^{1-a}\Big(H + \frac{a-1}{2}\Big) - iE_+^a\,\bar E_+^{-a}\Big(\bar H - \frac{a}{2}\Big)\,,$$

$$p_\gamma = \frac{\ell}{\hbar}\,(E_+ - \bar E_+), \quad \gamma = -iE_+^{a-1}\bar E_+^{1-a}\Big(H + \frac{a-1}{2}\Big) + iE_+^a\,\bar E_+^{-a}\Big(\bar H - \frac{a}{2}\Big)\,.$$

Coordinates, momenta

- momenta form a Lie algebra

$$[p_z, p_\gamma] = \frac{1}{\ell}\, p_\gamma, \quad [p_z, p_\beta] = \frac{1}{\ell}\, p_\beta, \quad [p_\beta, p_\gamma] = 0$$

- $(\beta \pm \gamma)/z$ are the lightcone coordinates on the boundary $z = 0$: boundary is a commutative flat plane:

$$\left[p_\beta + p_\gamma, \, \frac{\beta+\gamma}{z}\right] = \frac{2}{\ell}\,, \quad \left[p_\beta - p_\gamma, \, \frac{\beta-\gamma}{z}\right] = \frac{2}{\ell}$$

- Poincaré coordinates are dimensionless, so commutative limit should and can be discussed for other, e.g. embedding coordinates

$$[y - v, u + x] = \frac{a\hbar}{\ell}\,(u + x)\, E_+^{-1}\ .$$

The frame 1-forms anticommute and the connection is Levi-Civita.

Connection:

$$\omega^z{}_\beta = \theta^\beta, \quad \omega^z{}_\gamma = -\theta^\gamma, \quad \omega^\beta{}_z = -\theta^\beta, \quad \omega^\gamma{}_z = -\theta^\gamma$$

Curvature:

$$\Omega^z{}_\gamma = \theta^z\theta^\gamma, \quad \Omega^z{}_\beta = -\theta^z\theta^\beta, \quad \Omega^\beta{}_z = \theta^z\theta^\beta, \quad \Omega^\beta{}_\gamma = \theta^\beta\theta^\gamma,$$
$$\Omega^\gamma{}_z = \theta^z\theta^\gamma, \quad \Omega^\gamma{}_\beta = \theta^\beta\theta^\gamma$$

These values coincide with the classical expressions,

$$R_{zz} = -2, \quad R_{\beta\beta} = -2, \quad R_{\gamma\gamma} = 2, \quad R_{ab} = -2g_{ab}$$

BTZ identifications

The advantage of complicated z, β and γ introduced before is that

$$[H-\bar{H}, z] = (2a-1)z, \ \ [H-\bar{H}, \beta+\gamma] = 2(a-1)(\beta+\gamma), \ \ [H-\bar{H}, \beta-\gamma] = 2a(\beta-\gamma)$$

Introducing $U = e^{\alpha(H-\bar{H})}$ we find that coordinates transform as

$$UzU^{-1} = e^{\alpha(2a-1)}z, \ U(\beta+\gamma)U^{-1} = e^{2\alpha(a-1)}(\beta+\gamma), \ U(\beta-\gamma)U^{-1} = e^{2\alpha a}(\beta-\gamma)$$

Transformations have the same form as classical BTZ identifications. By demanding that they coincide we get unique solution for α and a,

$$\alpha = -\frac{2\pi r_-}{\ell}, \quad a = \frac{r_+ + r_-}{2r_-}.$$

Each choice of parameters α and a gives a different BTZ space with the corresponding $r_\pm$. For $a = 1/2$, $\alpha = -2\pi i$ we formally obtain AdS_3.

In order to analyse coordinates we represent fuzzy BTZ black hole in (Fourier-transformed) Hilbert space representation $\mathcal{H}\otimes\bar{\mathcal{H}}$, $\mathcal{H}\cong T^-_l$, $\bar{\mathcal{H}}\cong T^-_{\bar{l}}$. Wave functions are functions of two real variables $x, \bar{x} > 0$. We have

$$p_z = x\partial_x + \bar{x}\partial_{\bar{x}} + l + \bar{l} + 2, \qquad z = 2x^a\,\bar{x}^{1-a}$$

$$p_\beta = -i(x+\bar{x}), \qquad \beta+\gamma = -2i\left(\frac{x}{\bar{x}}\right)^{a-1}\left(x\partial_x + l + \frac{a+1}{2}\right)$$

$$p_\gamma = -i(x-\bar{x}), \qquad \beta-\gamma = -2i\left(\frac{x}{\bar{x}}\right)^{a}\left(\bar{x}\partial_{\bar{x}} + \bar{l} + 1 - \frac{a}{2}\right)$$

U acts on $f(x,\bar{x})$ as $(Uf)(x,\bar{x}) = e^{\alpha(l-\bar{l})} f\,(e^{\alpha}x, e^{-\alpha}\bar{x})$

Coordinate change

We introduce new coordinates $\chi \in (0, \infty)$ and $\eta \in (-\infty, \infty)$ by

$$x = \chi e^{\eta}, \qquad \bar{x} = \chi e^{-\eta} \ .$$

For $l = \bar{l}$ we find that U acts only on η, as a finite translation

$$H - \bar{H} = x\partial_x - \bar{x}\partial_{\bar{x}} = \partial_\eta \, , \qquad (Uf)(\chi, \eta) = f(\chi, \eta + \alpha) \ .$$

We require invariance of the BTZ wave functions under the discrete subgroup generated by U assuming that η is periodic

$$\eta \sim \eta + \alpha n, \quad n \in \mathbb{Z}$$

The radial coordinate r is given by

$$B(r) = \frac{r^2 - r_+^2}{r_+^2 - r_-^2} = \frac{\beta^2 - \gamma^2}{z^2} \geq -\frac{r_+^2}{r_+^2 - r_-^2}$$

$\ell = 1$. The last inequality is the condition $r^2 \geq 0$.

In the given representation, after symmetric ordering we obtain

$$B = \frac{1}{4}\Big(-\partial_\chi^2 - \frac{4l+3}{\chi}\,\partial_\chi - \frac{(2l+1)^2}{\chi^2} + \frac{1}{\chi^2}\,\partial_\eta^2\Big)$$

and the Ansatz for the eigenfunctions is

$$f_{n,\lambda}(\chi,\eta) = e^{\frac{2\pi i n}{\alpha}\,\eta}\, f_\lambda(\chi) = e^{\frac{2\pi i n}{\alpha}\,\eta}\,\chi^{-2l-\frac{3}{2}}\, h_\lambda(\chi)\,, \quad n \in \mathbb{Z}\ .$$

r-spectrum

The eigenvalue equation, written in the Schrödinger form, becomes

$$-\frac{d^2 h_\lambda}{d\chi^2} - (c^2 + \frac{1}{4})\,\frac{1}{\chi^2}\,h_\lambda = 4\lambda^2\,h_\lambda\,, \quad c = -\frac{2\pi n}{\alpha} = \frac{n\,\ell}{r_-}\ .$$

It is the equation for a particle in the attractive χ^{-2} potential.

$$V(\chi) = -(c^2 + \frac{1}{4})\,\frac{1}{\chi^2}\ .$$

This quantum-mechanical potential is very specific.

Solutions are the Bessel functions: for scattering states $\lambda^2 > 0$

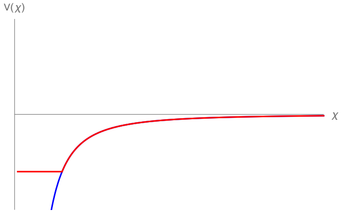
$$h_\lambda(\chi) = \sqrt{\chi}\,H^{(1,2)}_{ic}(2\lambda\chi) \sim \frac{1 \pm i}{\sqrt{2\pi\lambda}}\ e^{\pm\left(2i\lambda\chi + \frac{c\pi}{2}\right)}\,, \quad \chi \to \infty$$

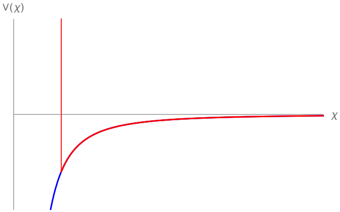
and for bound states $\lambda^2 < 0$, $\kappa^2 = -\lambda^2$

$$h_\lambda(\chi) = \sqrt{\chi}\,H^{(1)}_{ic}(2i\lambda\chi) = \sqrt{\chi}\,K_{ic}(2\kappa\chi) \sim \frac{1}{2}\sqrt{\frac{\pi}{\kappa}}\ e^{-2\kappa\chi}\,, \quad \chi \to \infty$$

The problem is that the equation is scale invariant. Technically, it arises because B is not a self-adjoint operator, and each choice of the self-adjoint extension gives a different spectrum of B.

In our case: we should in addition ensure that $r^2 \geq 0$ holds. In consequence, eigenvalues of B are to be constrained as $\lambda^2 \geq -\frac{r_+^2}{r_+^2 - r_-^2}$.





We obtain a discrete infinite set of bound states whose eigenvalues exponentially accumulate at $\lambda^2 \to 0-$.

Summary

- we presented a model of noncommutative black hole in 3d constructed using the spacetime symmetries
- quantisation of geometry is good: metric is the same as classical, curvature is constant, noncommutative Einstein space
- BTZ boundary is 2d flat space
- BTZ identifications are realised on the quantum level: as ∘ transformation of coordinates, ∘ restriction of variable η, similar to ϕ in the classical BTZ
- the outer horizon r_+ appears naturally, as dividing point between continuous and discrete parts of the spectrum of radial coordinate
- discreteness of the below-horizon states

Open questions

- the role of r_-?
- BTZ identifications are singular for $r_- = 0$: description of non-rotating BTZ?
- from representation theory we see that noncommutative limit is not correct if we compare algebras of functions: besides the scalar there are higher spin modes?
- is there a geodesic singularity? motion of particles?
- extension of r, α and a to complex values: a wormhole?
- # of states below horizon and entropy: a further regularisation?
- etc.